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SOLUTION OF THE INVERSE BOUNDARY-VALUE PROBLEM OF THE THEORY OF SHELLS

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A numerical solution of the boundary-value problem of the theory of shells with an unknown free boundary is performed using Galerkin's method. An analytical solution of the inverse problem is obtained using the potential theory method. The calculation results for the unknown domain of solution of the Helmholtz type basic equation for a spherical shell that is unclosed in two coordinates are compared with the calculation data of the finite-element method.

The linear theory of plates and cylindrical shells [1–4] is reduced to the solution of Helmholtz's equation; therefore, the obtained solutions can be used, in particular, for selecting the shape of holes (or reinforcements) in shells of rotation that provides minimum stress concentration [5].

The numerical solution of boundary-value problems with an unknown free boundary involves some computational difficulties [6–8]; in this paper we show the possibility of simplifying numerical solution by using an analytical approximate method.

Solutions of the inverse problems of the theory of elasticity are presented in [9, 10]. The maximum size of the domain of solution of the hyperbolic equation of a toroidal shell was determined in [11] using the method of characteristics. A similar problem is solved in this paper for an elliptic domain.

1. Let us consider a segment of a spherical shell that is bounded by the coordinates $0 \leqslant \varphi \lesssim 2\pi$ and $\theta_1 \leqslant \theta \leqslant \theta_2$ (φ and θ are the angular coordinates in the circumferential and meridional directions). The force P_z is applied through a moving rigid disc to the edge θ_1 and the edge θ_2 is rigidly fixed and remains immovable (Fig. 1). The edges $\varphi = 0$ and $\varphi \approx 2\pi$ are free from external stresses.

The zone of influence of the edge $\varphi = 0$ (which determines the domain of solution of the basic equation of the shell) is unknown and is found using the method of integral equations. Since the moment stressed state of the shell changes faster than the tensile stresses, we use the moment-free theory of shells to determine the unknown domain of solution. The basic equation has the form [12]

$$\frac{1}{R_1 R_2 \sin\theta}\frac{\partial}{\partial\theta}\left[\frac{R_2^2 \sin\theta}{R_1}\frac{\partial U}{\partial\theta}\right] + \frac{1}{R_2 \sin^2\theta}\frac{\partial^2 U}{\partial\varphi^2} = 0, \tag{1.1}$$

where $U = -T_2 R_1 \sin^2\theta$; T_2 is the tensile stress in the circumferential direction; R_1 and R_2 are the radii of the main curvatures of the median surface in the meridional and circumferential directions.

For small angles θ, assuming that $\sin\theta \approx \theta$ and $\cos\theta \approx 1$, we transform (1.1) into the equation

$$\frac{\partial^2 W}{\partial\beta^2} + \frac{\partial^2 W}{\partial\varphi^2} - \frac{9}{4}W = 0, \quad W = T\,\mathrm{e}^{5\beta/2}, \quad \beta = \ln\theta. \tag{1.2}$$

We specify the boundary conditions with allowance for the continuity and axial symmetry of the stressed state outside of the zone of influence of the free edge $\varphi = 0$:

$$W\Big|_{\varphi=0} = 0, \quad W\Big|_L = f(\beta), \quad \frac{\partial W}{\partial\varphi}\Big|_L = 0. \tag{1.3}$$

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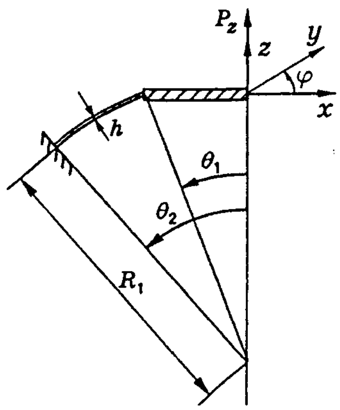

Fig. 1

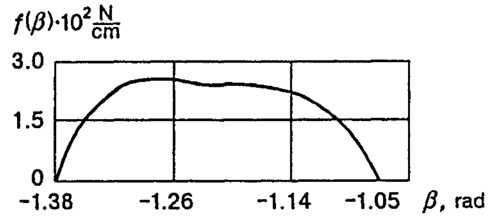

Fig. 2

Here L is the unknown domain boundary of solution of Eq. (1.1); $f(\beta)$ is a function that is known from the calculation of the shell closed in φ. Figure 2 shows the function $f(\beta)$ that is calculated [13] for shell parameters $0.25 \leqslant \theta \leqslant 0.35$ rad, $R_1 = 382$ mm, $h = 0.1$ mm, $E = 2.1 \cdot 10$ MPa (modulus of elasticity), $\mu = 0.3$ (Poisson's ratio), and $P = 9.8$ N using the standard computer program for shells.

Introducing the function $\Phi(\beta, \varphi)$, we transform Eq. (1.2) into the form

$$\frac{\partial^2 \Phi}{\partial \beta^2} + \frac{\partial^2 \Phi}{\partial \varphi^2} - \frac{9}{4}\Phi = F(\beta), \quad \Phi(\beta, \varphi) = f(\beta) - W(\beta, \varphi), \quad F(\beta) = \frac{d^2 f}{d\beta^2} - \frac{9}{4} f. \tag{1.4}$$

We approximate the domain of solution of Eq. (1.4) by a set of rectangular elements with width d_j and height a_j ($j = 0, 1, 2, 3, \ldots, n$) (Fig. 3). To the left of the cross section I–I in Fig. 3 there is a semi-infinite band $j = 0$. The unknown part of the boundary of the domain of solution of (1.4) is a stepwise broken line L.

We write the boundary conditions (1.3) for Eq. (1.4), taking into account the change of variables:

$$\Phi\Big|_{\varphi=0} = f(\beta), \quad \Phi\Big|_{L} = 0; \tag{1.5}$$

at the sides of the rectangular elements parallel to the φ-axis, we set

$$\frac{d\Phi}{d\varphi} = 0. \tag{1.6}$$

In mathematical description, the boundary-value problem (1.4)–(1.6) is analogous to the problem of propagation of electromagnetic waves in a closed system of rectangular waveguides. A source of disturbance is assigned at the boundary a_0 of the first ($j = 1$) rectangle in the waveguide cross section, which is shown by a broken line in Fig. 3. The propagation and reflection of waves in the φ-direction are considered [14].

The solution of the boundary-value problem (1.4)–(1.6) for each element has the form [14–16]

$$\Phi(x, \varphi) = \sum_{n=0}^{\infty} F_m^j \, \frac{\sin \dfrac{n\pi}{a_j}\left(x + \dfrac{a_j}{2}\right)}{\gamma_l \sinh \gamma_l d_j}; \tag{1.7}$$

$$F_m^j = \frac{\varepsilon_n}{2a_j} \int\limits_{G(x,\varphi)}\!\!\int S_j(x) \sin \frac{n\pi}{a_j}\Big(x + \frac{a_j}{2}\Big) \eta \, dx \, d\varphi, \tag{1.8}$$

where x is a new variable chosen so that the φ-axis is the symmetry axis of the rectangular elements;

$$\gamma_l = \left[\left(\frac{n\pi}{a_l}\right)^2 - r^2\right]; \qquad r^2 = -\frac{9}{4} \quad (l = 0, 1, 2, \ldots, n);$$

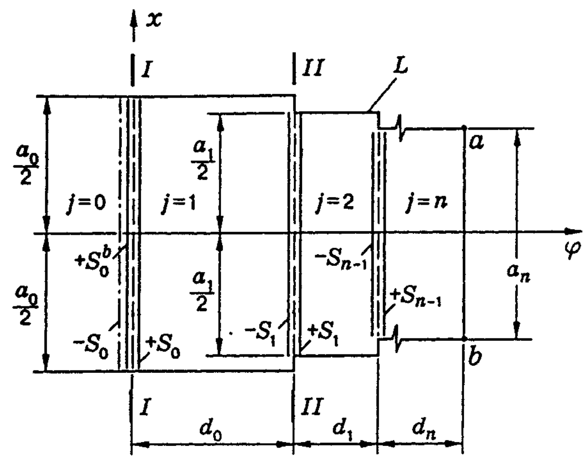

Fig. 3

the Neumann number is

$$\varepsilon_n = \begin{cases} 1 & \text{when } n = 0, \\ 2 & \text{when } n \neq 0; \end{cases}$$

the Green's function is

$$S_j(x) = \frac{d^2 f_{0j}(x)}{dx^2} + r^2 f_{0j}(x), \quad f_{0j}(x) = \sum_{\alpha=1}^{m} B_{j\alpha} \sin \frac{\alpha\pi}{a_j}\left(x + \frac{a_j}{2}\right); \tag{1.9}$$

$B_{j\alpha}$ are unknown coefficients;

$$\eta = \begin{cases} \cosh \gamma_l(d - \widehat{\varphi}) \cosh \gamma_l \varphi^0 & \text{when } \varphi^0 < \widehat{\varphi}, \\ \cosh \gamma_l \widehat{\varphi} \cosh \gamma_l(d - \varphi^0) & \text{when } \varphi^0 > \widehat{\varphi}; \end{cases}$$

$\widehat{\varphi}$ is the conjunction coordinate of the rectangular elements; φ^0 is the coordinate of the integration plane of the influence function which characterizes the electromagnetic fields in individual rectangular elements. The function $\eta(\varphi^0, \widehat{\varphi})$ is found on the basis of the theorem of equivalence; the actual electromagnetic field sources are replaced by the equivalent surface currents, and integral equations of the type (1.8) are formulated [16].

In the cross section I-I (Fig. 3) the solution of (1.7) has the form

$$\Phi(x, n) = \frac{1}{\gamma_0} \sum_{n=0}^{\infty} F_m^{(0)} \sin \frac{n\pi}{a_0}\left(x + \frac{a_0}{2}\right) e^{\pm\gamma_l \varphi}; \tag{1.10}$$

$$F_m^{(0)} = \frac{\varepsilon_n}{2a_0} \int\int_{G(x,\varphi)} \tilde{S}_0(x) \sin \frac{n\pi}{a_0}\left(x + \frac{a_0}{2}\right) e^{\pm\gamma_l \varphi}. \tag{1.11}$$

Here $\tilde{S}_0(x) = S_j(x) + S_0^b(x)$; $S_0^b(x)$ is determined by the expression

$$S_0^b(x) = \frac{d^2 f_0^b(x)}{dx^2} + r^2 f_0^b(x); \qquad f_0^b(x) = \sum_{i=1}^{3} A_i \sin \frac{i\pi}{a_0}(x + \frac{a_0}{2});$$

$f_0(x)$ is the expansion of the function $f(\beta)$ given in the boundary conditions (1.5) in terms of sines; the upper and lower signs in the exponent refer to $\varphi < \widehat{\varphi}$ and $\varphi > \widehat{\varphi}$, respectively.

From the conjunction conditions of solutions (1.7), for each rectangular element in the cross sections I-I and II-II, we have the following system of equations (the other equations are of similar form and are

TABLE 1

Circumferential coordinate, rad	$\sigma_1 = T_1/h$	$\sigma_2 = T_2/h$
	MPa	
0.196	−0.485	4.94
0.392	−0.729	5.058
0.588	−0.805	5.082
0.794	−0.829	5.089
0.98	−0.831	5.096
1.078	−0.831	5.098

therefore not presented):

$$\begin{aligned} &\sum_{n=0}^{\infty} \Phi_{j=0}\Big|_{\varphi=0-\Delta} = \sum_{n=0}^{\infty} \Phi_{j=1}\Big|_{\varphi=0+\Delta} \qquad \text{(cross section I–I)},\\ &\sum_{n=0}^{\infty} \Phi_{j=1}\Big|_{\varphi=d_0-\Delta} = \sum_{n=0}^{\infty} \Phi_{j=2}\Big|_{\varphi=d_0+\Delta} \qquad \text{(cross section II–II)} \end{aligned} \tag{1.12}$$

(Δ is an infinitesimal small increment of the coordinate φ).

With given a_j and d_j, after integrating Eqs. (1.12), we obtain a system of algebraic equations from which the coefficients $B_{j\alpha}$ are found.

The first boundary condition (1.5) is the source of disturbance. The minimum condition of the field of a "reflected wave" determines the zone of influence of the source and corresponds to the minimum of the integral J_0:

$$J_0 = \int_{-a_0/2}^{a_0/2} \left[\frac{d^2}{dx^2}(B_{0\alpha}\xi) + r^2 B_{0\alpha}\xi\right] dx, \qquad \xi = \sin\frac{\alpha\pi}{a_0}\left(x + \frac{a_0}{2}\right).$$

Setting a successive set of a_j and d_j values beginning from the maximum possible values and performing their discrete reduction with step $\overline{\Delta} = 0.01$, we calculate the integral J_0. From the condition of minimum of J_0 we find that the maximum size of the solution domain φ_0 for the given shell parameters is $\varphi_0 = 0.74$ rad.

As follows from (1.10), when r^2 is negative the solution decays exponentially in the φ direction. In this case the derivative $d\Phi/d\varphi$ at the part (ab) of the boundary (Fig. 3) is practically equal to zero, which corresponds to the condition of axial symmetry and continuity of the stressed state of the shell of rotation outside of the zone of influence of the free edge $\varphi = 0$. The exponent also depends on the number of terms of the series (1.10). It follows from the calculations that the series (1.10) decreases rapidly with increasing number of terms, and only the first three terms can be used with sufficient accuracy for practical applications.

Calculations showed that a 2-fold increase in the function $f(\beta)$ (without change in its form) in the first boundary condition (1.5) produces only a 14% increase in the maximum size of the unknown domain of solution. Calculations for a spherical shell with the above parameters and the same loading conditions were carried out using the finite-element method. The numerical calculation results, i.e., the tensile stresses σ_1 and σ_2 in the central part of the segment of the spherical shell, are given as a function of the coordinate φ in Table 1.

As follows from Table 1, the zone of influence of the free edge $\varphi = 0$ is bounded by the coordinate $\varphi_0^{(1)} = 1.03$ rad.

2. We find the unknown domain of solution of Eq. (1.1) by hydrodynamic analogy. The calculation of the stressed state of the shell is reduced to the Laplace equation, and the theory of axisymmetric potential flows is also determined by this equation. The possibility of applying hydrodynamic analogy to the solution of problems of the theory of elasticity was supported experimentally by the works of R. Bowde [17, p. 88].

For a segment of a spherical shell for which $R_1 = R_2$, we have $\sin\theta \approx \theta$ and $\cos\theta = 1$. Then Eq. (1.1)

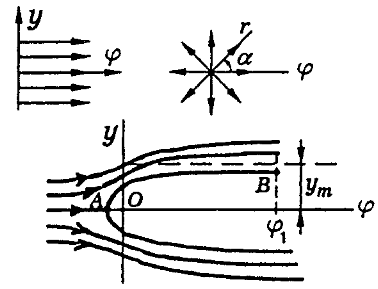

Fig. 4

becomes

$$\theta^2\frac{\partial^2 U}{\partial\theta^2}+\theta\frac{\partial U}{\partial\theta}+\frac{\partial^2 U}{\partial\varphi^2}=0. \tag{2.1}$$

Introducing the change $\theta = e^y$, from (2.1) we obtain

$$\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial\varphi^2}=0. \tag{2.2}$$

Let us consider the case of superposition of two potential flows that are calculated using the Laplace equation: uniform rectilinear flow and flow with a "point" radial source. The velocity of effusion from the "point" source is $\overline{U}_r = m/r$ (the constant m characterizes the effusion intensity and r is a coordinate). The complex potential of the total flow is written as [18]

$$\omega = \overline{\Phi} + i\psi = -\overline{U}z - m\log z, \tag{2.3}$$

where $\overline{\Phi}$ is the velocity potential, ψ is the stream function, $\overline{U}$ is the velocity of the laminar rectilinear flow; $z = \varphi + iy = r^{i\alpha}$ (Fig. 4). The derivative of Eq. (2.3) is $d\omega/dz = -\overline{U} - m/z$.

The critical point at which the velocity of the total flow equals zero is found as the roots of the equation $d\omega/dz = 0$ and the equality $z = -m/\overline{U}$ determines the point at which the velocities of two flows are equal (point A in Fig. 4).

The stream function is of the form [18]

$$\psi = -\overline{U}y - m\alpha = -\overline{U}y - m\arctan\frac{y}{\varphi}. \tag{2.4}$$

By virtue of the symmetry with respect to the φ-axis we consider only half of the total flow for $y \geqslant 0$. The angle α is reckoned from the positive direction of the φ-axis (Fig. 4), on the negative part of the φ-axis for $\alpha = \pi$ $y = 0$, and for this part of the total flow it follows from (2.4) that $\psi = -m\pi$ and the equation of the streamline branching at point A is representable as (Fig. 4)

$$-\pi m = -\overline{U}y - m\alpha. \tag{2.5}$$

Equation (2.5) involves the negative part of the φ-axis up to point A and curve AB (Fig. 4). From (2.5), we see that for $\alpha \to 0$ the coordinate $y \to m\pi/\overline{U} = y_m$, i.e., curve AB is bounded by the asymptote $y_m =$ const. In view of the symmetry of the streamline AB with respect to the φ-axis there is a second asymptote $y = -y_m$. Knowing the coordinate $z = -m/\overline{U}$ of the critical point A, from (2.4), we find OA$= y_m/\pi$ and the equation of curve AB ($\psi =$ const)

$$\frac{1}{\varphi} = -\frac{1}{y}\tan\frac{\pi y}{y_m}. \tag{2.6}$$

It follows from (2.6) that $y_1/y_m = 6.32$ when $y/y_m = 0.95$.

3. Conclusions.

(1) From the numerical solution by Galerkin's method it is obvious that the domain of solution of Eq. (1.2) is bounded by an oval contour with a ratio of semi-axes of 4.48 : 1, while according to the data of calculation of the sector of the spherical shell by the finite-elements method this ratio is 6.24 : 1. This difference can be explained by the fact that the basic equation (1.2) is obtained following the approximate moment-free theory, while the finite-element method takes into account bending moments, and also by the fact that the approximation of the unknown domain of solution (1.2) by the system of rectangular elements and the singularities of conjunction of the solutions for individual rectangular elements at angular points introduce a systematic error in numerical realization of the algorithm. An increase in the number of rectangles does not eliminate this error. Therefore, the results of approximate analytical solution by means of hydrodynamic analogy, in which the unknown boundary is given by a smooth curve, are closer to the calculation results using the finite-element method, and an estimate of the maximum size of the unknown domain of solution can be obtained from Eq. (2.6).

(2) A 2–3-fold increase in the boundary value of the function $f(\beta)$ in boundary condition (1.5), which characterizes the influence of the external load, produces, respectively, a 14% and 16% increase in the solution domain. These results are confirmed in [19], in which a plate on an elastic foundation with external force P applied to the center of the plate is considered and an increase in the force P is shown to bring about no change in the domain of its influence.

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